Theoretical Models of Vote Buying: A Selective Guided Tour

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Part 3

SEQUENTIAL COMPETITIVE VOTE BUYING WITH PROCEDURAL LEGISLATORS

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Introductory Remarks (I)

- This is the model that I have explored in my most recent research together with Vera Zaporozhets and also Peter Sudholter.
- The two lobbyists move in sequence. We eliminate the existence difficulties.
- Cournot versus Stackelberg.
- The result also tells us how much asymmetry is needed to restore an equilibrium in pure strategies in the Colonel Blotto Game.
- The model has been pionnered by Groseclose and Snyder (1996)
- It has been used by many authors including Banks (2000), Dieiermer and Myerson (1999) and Morgan and Vardy (2007)(2008).
- Alternative sequential Models (Dekel, Jackson and Wolinski (2008)

Introductory Remarks (II)

Key Parameters of the Environment

- The maximal willingness to pay for winning (i.e. to have their favorite policy selected) of the two lobbies. These two numbers represent the economic stakes under dispute and determine the intensity and asymmetry of the competition.
- The voting rule describing the legislative process.
- The heterogeneity in the legislators' preferences.
- The binary setting considered in this paper is the simplest setting to tackle the joint influence of these three inputs on the final outputs.

Introductory Remarks (III)

Items (I)

- The first item consists of a single number per lobby, i.e. the amount of money this lobby is willing (able) to invest in this competition.
- The second item is also very simple. In this simplistic institutional setting, with no room for agenda setting or other sophisticated legislative action which would arise in the case of a large multiplicity of issues, we only need to know the winning coalitions, i.e. the coalitions of legislators able to impose the reform if the coalition unanimously supports this choice. Despite its apparent simplicity, this combinatorial object allows accommodating a wide diversity of legislatures. Banks and Groseclose and Snyder focus on the standard majority game, while Diermeier and Myerson consider the general case as we do.

Introductory Remarks (IV)

Items (II)

- The third item describes the differences between the legislators other than those already attached to the preceding item if these legislators are not equally powerful or influential in the voting process. This "second" heterogeneity dimension refers to the differences between their intrinsic preferences for the reform versus the status quo. This difference, measured in monetary units, can be large or small and negative or positive. Diermeier and Myerson disregard this dimension by assuming that legislators are indifferent between the two policies, while Banks and Groseclose and Snyder consider the general situation but derive their results under some specific assumptions. We assume that legislators unanimously prefer the reform to the status quo but differ with respect to the intensity of their preference.

Introductory Remarks (V)

Contribution 1 (I)

- **The first contribution consists in identifying the conditions under which the lobby moving first will make positive offers to some legislators.**
- In this sequential game, the lobby moving last has an advantage as it can react optimally to its the opponent's offers without fear of response. If the asymmetry is too weak, lobby 1 will abandon the prospect of influencing the legislature as it will be rationally anticipating its defeat; in fact, it will make offers only if it is certain of success. If it does not make any offer, it is enough for lobby 0 to compensate a minimal winning coalition of legislators for their intrinsic preferences towards reform. Lobby 1 will participate if its willingness to pay or budget is larger than lobby 0's willingness to pay or budget. This minimal amount of asymmetry, which we call the *victory threshold*, defines by how much lobby 1's stake must outweigh lobby 0's stake to

Introductory Remarks (VI)

Contribution 1 (II)

- Our first result states that the calculation of the victory threshold amounts to calculating the supremum of a linear form over a convex polytope, which is closely related to the polytope of balanced families of coalitions introduced in cooperative game theory to study the core and other solutions. This result enables us to take advantage of the voluminous amount of work which has been done on the description of balanced collections. When heterogeneity in legislators' preferences is ignored, the victory threshold only depends upon the simple game describing the rules of the legislature. **It corresponds to what Diermeier and Myerson have called the hurdle factor of the legislature**. Quite surprisingly, this single parameter acts a summary statistic allowing us to predict the minimal budget that lobby 1 needs to invest to win the game.

Introductory Remarks (VII)

Contribution 2

- The second contribution consists in connecting the problem of computing the hurdle factor to the *covering problem*, which is one of the most famous, but also most difficult problems in the combinatorics of sets or hypergraphs. We establish the connection with another famous parameter of a simple game, and calculate the hurdle factor of several simple games. Once, it is established that the hurdle factor is the fractional covering number of a specific hypergraph, we can take advantage of the enormous body of knowledge in that area of combinatorics.

Introductory Remarks (VIII)

Contribution 3

- The third contribution consists in showing that the hurdle factor can alternatively be calculated, surprisingly, as the maximum of specific equity criteria over the set of imputations of a cooperative game with transferable utility attached to the simple game of the legislature. The specific equity criterion is the minimum, across coalitions, of what the members of the coalitions will get in the imputation and what they could get on their own, i.e. the first component in the lexicographic order supporting *the nucleolus*. We use that result to show how to calculate the hurdle factor for the important class of weighted majority games. While there is a link between the weights of the legislators and the hurdle factor when the game is homogeneous, we show that the relation is more intricate in the general case.

Introductory Remarks (IX)

How much costs a legislator ?

- The connection with cooperative game theory is very surprising. The first results tell us the size of the lobbying budget is the hurdle factor times the willingness to pay (or budget) of lobby 0. It remains to understand how this budget will be allocated across legislators. This is, of course, an important question as we would like to understand which legislator's characteristics determine lobby 1's willingness to buy his support and the amount that he will receive for selling his vote.
- As already discussed, legislators differ in two respects: the intensity of their preference for lobby 1 and their position/power in the legislature. The price of a legislator's vote is likely to be a function of both parameters. We show that the set of equilibrium offers is the least core of the cooperative game used to calculate the hurdle factor. It may contain multiple solutions, but the nucleolus is always one of

Introductory Remarks (X)

Power Measurement

- One important conclusion is that these prices have often little to do with a legislator's power as calculated through either the Banzhaf index (Banzhaf (1962), (1968)) or the Shapley-Shubik index (Shapley and Shubik (1954)). This suggests that the axiomatic theory of power measurement may not be fully relevant to predict players' payoffs in a game like this one.

Ingredients (I)

Lobbies

- The external forces that seek to influence the legislature are represented by two players, whom we call lobby 0 and lobby 1.
- Lobby 1 wants the legislature to pass a bill (change, proposal, reform) that would change some area of law. Lobby 0 is opposed to this bill and wants to maintain the status quo.
- Lobby 0 is willing to spend up to W_0 dollars to prevent passage of the bill, while lobby 1 is willing to pay up to W_1 dollars to pass the bill. Sometimes, we refer to these two policies in competition as being policies 0 and 1. We assume that $\Delta W \equiv W_1 - W_0 > 0$.

Ingredients (II)

Legislature (I)

- The legislature is described by a *simple game* i.e. a pair $(N, \mathcal{W})$, where $N = \{1, 2, ..., n\}$ is the set of legislators and $\mathcal{W}$ the set of *winning* coalitions, satisfies: $S \in \mathcal{W}$ and $S \subseteq T$ implies $T \in \mathcal{W}$. The interpretation is the following. A bill is adopted if and only if the subset of legislators who voted for the bill forms a winning coalition. From that perspective, the set of winning coalitions describes the rules operating in the legislature to make decisions.
- A coalition C is *blocking* if $N \backslash C$ is not winning: some legislators (at least one) from C are needed to form a winning coalition. We will denote by $\mathcal{B}$ the subset of blocking coalitions; by definition, the status quo is maintained as soon as the set of legislators who voted against the bill forms a blocking coalition.

Ingredients (III)

Legislators

- All legislators are assumed to be biased towards policy 1, i.e. all of them will vote for policy 1 against policy 0 if no other event interferes with the voting process.In contrast to Banks (2000) and Groseclose and Snyder (1996), our assumption on the preferences of legislators rules out the existence of horizontal heterogeneity. However, legislators also value money, so we introduce instead some form of vertical heterogeneity.
- Specifically, we assume that legislators differ with regard to their willingness to depart from social welfare. The type of legislator i, denoted by α_i, is the minimal amount of dollars that he needs to receive in order to sacrifice one dollar of social welfare. Therefore, if the policy adopted generates a level of social welfare equal to W, legislator i's payoff if he receives a transfer t_i is:

$$t_i + \alpha_i W$$

Ingredients (IV)

Legislators

- This payoff formulation is compatible with two behavioral assumptions.
- A first possibility is that the component W appears as soon as the legislator has voted for a policy generating a level of social welfare W regardless of whether this policy has ultimately been selected: we will refer to this model as *behavioral model P*, where P stands for procedural.
- Alternatively, we may assume that the component W appears whenever the policy ultimately selected generates a level of social welfare W regardless of whether the legislator has voted for or against this policy: we will refer to this model as *behavioral model C*, where C stands for consequential.

Strategies, Timing and Information (I)

- To promote passage of the bill, lobby 1 can promise to pay money to individual legislators conditional(we assume open voting) on their supporting the bill. Similarly, lobby 0 can promise to pay money to individual legislators conditional on their opposing the bill. We denote by $t_{i0} \geq 0$ and $t_{i1} \geq 0$ the (conditional) offers made to legislator i by lobbies 0 and 1 respectively. The corresponding n-dimensional vectors will be denoted by t_0 and t_1 respectively.
- The timing of actions and events that we consider to describe the lobbying game is the following.

1. Nature determines the type of each legislator.
2. Lobby 1 make contingent monetary offers to individual legislators.

Strategies, Timing and Information (II)

3. Lobby 0 observes the offers made by lobby 1 and makes contingent monetary offers to individual legislators
4. Legislators vote.
5. Payments (if any) are implemented.

Strategies, Timing and Information (III)

- This game has $n+2$ players. A strategy for a lobby is a vector in $\Re^n_+$. Each legislator can choose among two (pure) strategies: to oppose or to support the bill. The important thing to note is that the two lobbies move in sequence.
- To complete the description of the game, we should specify the information held by the players when they act. We focus here on the case where both the vector(W_0, W_1) of Lobbies' types and the vector $\alpha = (\alpha_1, \alpha_2, ..., \alpha_n)$ of legislators' types is common knowledge and without loss of generality such that $\alpha_1 \leq \alpha_2 \leq ... \leq \alpha_n$.. We refer to this informational environment as *political certainty*.

Strategies, Timing and Information (IV)

- The lobbies know the types of legislators when they make their offers and second, each legislator knows the type of any other legislator when voting. The environment where the type α_i of legislator i is private information, to which we refer as *political uncertainty*, is analyzed in Le Breton and Zaporozhets (2007).
- We examine the *subgame perfect Nash equilibria* of the game. They are called hereafter equilibria.

What do we want to know ?

- Under which conditions vote buying takes place i.e. when is it the case that lobby 1 makes an offer ?
- When an offer is made, how large is the group of legislators receiving an offer : minimal winning versus supermajorities ?
- How the aggregate vote buying budget of lobby 1 is allocated accross legislators : Who are the legislators receiving offers (clientelism versus swing legislators) and how the price of a legislator depends upon his/her status in the legislature and its ex-ante partisanship/opposition ?

The Victory Threshold (I)

- Our first objective is to calculate a key parameter of the game $v^*(\mathcal{B}, \boldsymbol{\alpha})$, which we call the *victory threshold*. Once calculated, this parameter leads to the following preliminary description of the equilibrium. Either the ratio $\frac{W_1}{W_0}$ is larger than or equal to the victory threshold and then lobby 1 makes an offer and wins the game, or $\frac{W_1}{W_0}$ is smaller than the victory threshold and then lobby1 does not make any offer and lobby 0 wins the game.
- The victory threshold depends both upon the vector of types α and the simple game $(N, \mathcal{W})$. Given the second-mover advantage, the victory threshold is larger than or equal to 1. Therefore,while necessary, $W_1 > W_0$ is not sufficient in general to guarantee the victory of lobby 1. The victory threshold provides the smallest value of the relative differential leading to such a victory.

The Victory Threshold (II)

- Let $t_1 = (t_{11}, t_{21}, ..., t_{n1}) \in \Re^n_+$ be lobby 1's offers. Lobby 0 will find it profitable to make a counter offer if there exists a blocking coalition S such that:

$$\sum_{i \in S} \left(t_{i1} + \alpha^i W_1\right) < \sum_{i \in S} \alpha^i W_0 + W_0.$$

- If lobby 1 wants to make an offer that cannot be cancelled by lobby 0, it must satisfy the list of inequalities:

$$\sum_{i \in S} \left(t_{i1} + \alpha^i \Delta W\right) \geq W_0 \text{ for all } S \in \mathcal{B}.$$

The Fundamental Linear Program

- The cheapest offers t_1 meeting these constraints are the solutions of the following linear program:

$$\underset{t_1}{Min} \sum_{i \in N} t_{i1}$$

$$\text{s.t.} \quad \sum_{i \in S} \left(t_{i1} + \alpha^i \Delta W \right) \geq W_0$$

$$\text{for all } S \in \mathcal{B}$$

$$\text{and } t_{i1} \geq 0 \text{ for all } i \in N.$$

- Lobby 1 will find it profitable to offer t_1^* if the optimal value to this linear program is less than W_1. It is then important to be able to compute this optimal value denoted $v^* (\mathcal{B}, \boldsymbol{\alpha})$ and called the victory threshold.

Balancedness and the Dual Linear Program

- The Dual Linear program is:

$$\underset{\delta}{Max} \sum_{S \in \mathcal{B}} \delta(S) \left[W_0 - \sum_{i \in S} \alpha^i \Delta W \right]$$

subject to the constraints

$$\sum_{S \in \mathcal{B}_i} \delta(S) \leq 1 \text{ for all } i \in N$$

$$\text{and } \delta(S) \geq 0 \text{ for all } S \in \mathcal{B}.$$

- A family of coalitions $\mathcal{C}$ is *(sub)balanced* if there exists a vector $\delta \in \Re^{\#\mathcal{C}}$, called *(sub)balancing coefficients*, such that:

$$\begin{aligned} \sum_{S \in \mathcal{C}_i} \delta(S) \quad &\leq \quad (=1) \text{ for all } i \in N \\ \text{and } \delta(S) \quad &\geq \quad 0 \text{ for all } S \in \mathcal{C}. \end{aligned}$$

Result 1

- Either (i) $W_1 \geq \sum_{S \in \mathcal{B}} \delta(S) \left[W_0 - \sum_{i \in S} \alpha^i \Delta W \right]$ for all vectors of subbalancing coefficients δ attached to $\mathcal{B}$. Then lobby 1 offers t_1^*, lobby 0 offers nothing and the bill is passed.
- Or (ii) $W_1 < \sum_{S \in \mathcal{B}} \delta(S) \left[W_0 - \sum_{i \in S} \alpha^i \Delta W \right]$ for at least one vector of subbalancing coefficients δ attached to $\mathcal{B}$. Then lobby 1 does not make any offer, lobby 0 offers t_0^* with $t_{i0}^* = \alpha^i \Delta W + \varepsilon$ for all $i \in S$ and $t_{i0}^* = 0$ otherwise where ε is an arbitrarily small positive number and S is any coalition such that $\sum_{i \in S} \alpha^i \Delta W = \underset{T \in \mathcal{B}}{Min} \sum_{i \in T} \alpha^i \Delta W$ and the bill is not passed.
- The victory threshold $v^*(\mathcal{B}, \mathbf{0})$ is therefore $\underset{\delta \in}{Sup} \sum_{S \in \mathcal{B}} \delta(S) \left[W_0 - \sum_{i \in S} \alpha^i \Delta W \right]$

The Hurdle Factor

- When $\alpha = 0$, the victory threshold is proportional to W_0 i.e. $v^*(\mathcal{B}, \mathbf{0}) = \gamma^*(\mathcal{B}) W_0$ where $\gamma^*(\mathcal{B}) \equiv v^*(\mathcal{B}, \mathbf{0}) W_0$, called hereafter the *hurdle factor* (Diermeier and Myerson (1999), is the value of the problem:

$$\underset{\delta}{Max} \sum_{S \in \mathcal{B}} \delta(S)$$

subject to the constraints

$$\sum_{S \in \mathcal{B}_i} \delta(S) \leq 1 \text{ for all } i \in N$$

and $\delta(S) \geq 0$ for all $S \in \mathcal{B}$.

- We have the inequality

$$v^* \left(\mathcal{B}, \boldsymbol{\alpha}\right) + \Delta W \sum_{i \in N} \alpha^i \geq W_0 \gamma^* \left(\mathcal{B}\right)$$

Example 1 : Majority Game with Three Legislators

$S \in \mathcal{B}_m$ iff $\#S = 2$ i.e. $S = \{1,2\}$, $\{1,3\}$ and $\{2,3\}$. The set of vectors of subbalancing coefficients is the polytope described by the set of extreme points

$$(0,0,0), (1,0,0), (0,1,0), (0,0,1), \left(\frac{1}{2},\frac{1}{2},0\right), \left(\frac{1}{2},0,\frac{1}{2}\right), \left(0,\frac{1}{2},\frac{1}{2}\right)$$

and $\left(\frac{1}{2},\frac{1}{2},\frac{1}{2}\right)$.

$$v^*(\mathcal{B},\alpha) = \ Sup\left(\begin{array}{c} W_0 - (\alpha_1+\alpha_2)\,\Delta W, \\ \frac{3W_0 - 2(\alpha_1+\alpha_2+\alpha_3)\Delta W}{2}, \ 0 \end{array} \right),$$

$$\gamma^*(\mathcal{B}) = \frac{3}{2}.$$

Example 2 : An Apex Game

$S \in \mathcal{B}_m$ iff $S = \{1,2\}$, $\{1,3\}$, $\{1,4\}$ or $\{2,3,4\}$. According to Shapley (1967), besides partitions, the minimal balanced families of coalitions are (up to permutations):

$$\{\{1,2,3\},\{1,2,4\},\{1,3,4\},\{2,3,4\}\},$$
$$\{\{1,2\}\{1,3\}\{1,4\}\{2,3,4\}\},$$
$$\{\{1,2\}\{1,3\}\{2,3\}\{4\}\},\{\{1,2\}\{1,3,4\}\{2,3,4\}\}$$

with the following respective vectors of balancing coefficients $\left(\frac{1}{3},\frac{1}{3},\frac{1}{3},\frac{1}{3}\right)$, $\left(\frac{1}{3},\frac{1}{3},\frac{1}{3},\frac{2}{3}\right)$, $\left(\frac{1}{2},\frac{1}{2},\frac{1}{2},1\right)$ and $\left(\frac{1}{2},\frac{1}{2},\frac{1}{2}\right)$.

$$v^*(\mathcal{B},\alpha) = Sup\left(\begin{array}{c}\frac{5W_0-3(\alpha_1+\alpha_2+\alpha_3+\alpha_4)\Delta W}{3}, \\ W_0-(\alpha_1+\alpha_2)\Delta W, 0\end{array}\right)$$

$$\gamma^*(\mathcal{B}) = \frac{5}{3}$$

and a Weighted Majority Game

- $S \in \mathcal{B}_m$ iff $S = \{1,2\}$or $\{1,3\}$. The set of vectors of subbalancing coefficients is the polytope described by the set of extreme points $(0,0)$, $(1,0)$, $(0,1)$

$$v^*(\mathcal{B}, \alpha) = Sup\left(W_0 - (\alpha_1 + \alpha_2)\,\Delta W,\ 0\right),$$

$$\gamma^*(\mathcal{B}) = 1$$

- $S \in \mathcal{B}_m$ iff $S = \{1,2\}$, $\{1,3\}$, $\{1,4,5\}$, $\{2,3,4\}$ or $\{2,3,5\}$. The geometry of the polytope becomes more intricate. We will demonstrate later, through a different technique, that when $\alpha = \mathbf{0}$, the relevant extreme point is the vector $\left(\frac{1}{5}, \frac{1}{5}, \frac{3}{5}, \frac{2}{5}, \frac{2}{5}\right)$ i.e.

$$\gamma^*(\mathcal{B}) = \frac{9}{5}$$

Connections with hypergraphs (I)

- The connection applies in the special case where $\alpha = 0$
- A *hypergraph* is an ordered pair $H = (N, \mathcal{H})$ where N is a finite set of n vertices and $\mathcal{H}$ is a collection of subsets of N called edges.
- The *rank* of H is the integer $r(H) \equiv Max\{\#E : E \in \mathcal{H}\}$. If every member of $\mathcal{H}$ has r elements, we call it $r-$uniform. An $r-$uniform hypergraph H is called $r-$partite if there exists a partition $\{N_k\}_{1 \leq k \leq K}$ of N such that $\#(N_k \cap E) = 1$ holds for all $E \in \mathcal{H}$ and all $k = 1, ...K$.
- Given an integer k, a hypergraph is $k-$*wise intersecting* if any of its k edges have a non-empty intersection; intersecting is used in place of 2-intersecting.
- Given an integer k, a k-cover of H is a vector $t \in \{0, 1, ..., k\}^n$ such that:

$$\sum_{i \in S} t_i \geq k \text{ for all } S \in \mathcal{H} \qquad ((1))$$

Connections with hypergraphs (II)

- A $1-$cover ($1-$matching) is simply called a cover (matching) of H. Note that a cover is simply a set T intersecting every edge of H i.e. $T \cap E \neq \varnothing$ for all $E \in \mathcal{H}$ while a matching is a collection of pairwise disjoint members of $\mathcal{H}$.
- A $k-$cover t^* minimizing $\sum_{i \in N} t_i$ subject to the constraints (1) is called an optimal $k-$cover and $\gamma_k^*(H) \equiv \sum_{i \in N} t_i^*$ is called the $k-$covering number. A $k-$matching δ^*maximizing $\sum_{S \subseteq N} \delta(S)$ is called an optimal $k-$matching and $\mu_k^*(H) \equiv \sum_{S \subseteq N} \delta^*(S)$ is called the $k-$matching number.
- When $k = 1$, $\gamma_1^*(H)$ is the minimum cardinality of the covers and is called the *covering number* of H while $\mu_1^*(H)$ is the maximum cardinality of a matching and is called the *matching number* of H.

Connections with hypergraphs (III)

Fractional Covers

- A *fractional cover* of H is a vector $t \in \Re^n$ such that:

$$\begin{aligned} \sum_{i \in S} t_i &\geq 1 \text{ for all } S \in \mathcal{H} \quad (1) \\ \text{and } t_i &\geq 0 \text{ for all } i \in N. \end{aligned}$$

Connections with hypergraphs (III)

Fractional Matchings

- A fractional matching of H is a vector $\delta \in \Re^{\#\mathcal{H}}$ such that:

$$\begin{aligned} \sum_{S \in \mathcal{H}_i} \delta(S) &\leq 1 \text{ for all } i \in N \quad (2) \\ \text{and } \delta(S) &\geq 0 \text{ for all } S \in \mathcal{H}. \end{aligned}$$

- A fractional cover t^* minimizing $\sum_{i \in N} t_i$ subject to the constraints (1) is called an optimal fractional cover and $\gamma^*(H) \equiv \sum_{i \in N} t_i^*$ is called the *fractional covering number*. A fractional matching δ^*maximizing $\sum_{S \subseteq N} \delta(S)$ subject to the constraint (2) is called an optimal fractional matching and $\mu^*(H) \equiv \sum_{S \subseteq N} \delta^*(S)$ is called the *fractional matching number*.

Connections with hypergraphs (IV)

- The hurdle factor of the simple game $(N, \mathcal{W})$ is the fractional covering number of $H = (N, \mathcal{B})$.
- If money is available in indivisible units, then the appropriate parameter becomes $\gamma^*_{W_0}(H)$ where the integer W_0 is the value of policy 0 for lobby 0 (when $\mathcal{C} = \mathcal{B}$ i.e. when lobby 0 is the follower) expressed in monetary units (linear programs with integer constraints)
- The case where $W_0 = 1$ is of particular interest, as it describes the situation where lobby 0 has a single money unit to spend in the process. The problem is now purely combinatorial: who should be the legislators on which lobby 1 should spend one unit to prevent lobby 0 from a targeting a single pivotal legislator. Hereafter, the integer $\gamma^*_1(H)$ will be called the *integral hurdle factor*.

Connections with hypergraphs (V)

If we invert the order of moves between the two lobbies, then the relevant simple game is the *dual game* $(N, \mathcal{B})$ and the corresponding hurdle factor, which we will call the *dual hurdle factor*, is the fractional covering number of $H = (N, \mathcal{W})$. The following developments apply equally to both hurdle factors and we will often use the symbol $\mathcal{H}$ without specifying whether $\mathcal{H} = \mathcal{B}$ or $\mathcal{H} = \mathcal{W}$. For an arbitrary hypergraph H, we have the inequalities:

$$\mu_1^*(H) \leq \frac{\mu_k^*(H)}{k} \leq \mu^*(H) = \gamma^*(H) \leq \frac{\gamma_k^*(H)}{k} \leq \gamma_1^*(H). \quad (3)$$

Connections with hypergraphs (V)

Uniform Hurdle Factor (I)

- What are the implications of assuming that all the legislators receiving an offer from a lobby receive the same offer (Morgan and Vardy (2007, 2008) refer to these offers as non-discriminatory vote buying).. This means that from the perspective of any one of the two lobbies, the population of legislators is partitionned into two groups : those who receive an offer from that lobby and those who dont. Let T_1 to denote the group of legislators receiving an offer from lobby 1 and let t_1 be the amount of the offer.

Connections with hypergraphs (V)

Uniform Hurdle Factor (II)

- Since this limitation applies equally to both lobby 0 and lobby 1, the cheapest offer t_1 meeting these constraints is solution of the following linear program:

$$\underset{(T_1, t_1) \in 2^N \times \Re_+}{Min} \quad t_1 \ \left(\# T_1 \right)$$

$$\text{subject to the constraints}$$

$$t_1 \left(\# S \right) \geq W_0 \text{ for all } S \in \mathcal{B}_m$$

$$S \cap T_1 \neq \varnothing \text{ for all } S \in \mathcal{B}_m \qquad (4)$$

Connections with hypergraphs (VI)

- The solution of the above problem is strongly connected to the solution of the covering problem. Since it is linear in W_0 , let $W_0 = 1$. First, we note immediately from the second set of constraints that the set T_1 must be a cover for the hypergraph $(N, \mathcal{B}_m)$. On the other hand, the tightest constraint in the first set of constraints are those attached to the smallest S in $\mathcal{B}_m$. We deduce then that :

$$t_1 = \frac{1}{\underset{S \in \mathcal{B}_m}{Min} \# S}$$

Connections with hypergraphs (VII)

- The problem of lobby 1 is then equivalent to the minimal covering problem. Using our notations, we deduce that the value of the above linear program with integer constraints, called hereafter the *uniform hurdle factor* and denoted $\gamma_u^*(\mathcal{B})$, is equal to:

$$\frac{\gamma_1^*(\mathcal{B}_m)}{\underset{S \in \mathcal{B}_m}{Min} \# S}$$

Connections with hypergraphs (VII)

covering number: min number of vertices to touch every edge

matching number: max number of pairwise disjoint edges

Qualified Majorities/minorities

- . $S \in \mathcal{H}$ iff $\#S = q$ where q is a fixed integer. In that case, it is easy to show that $\gamma^*(\mathcal{H}) = \frac{n}{q}$. For instance, in the case of the winning coalitions of the majority game ($q = \frac{n+1}{2}$ is n is odd and $q = \frac{n+2}{2}$ is n is even), we obtain:

$$\gamma^*(\mathcal{H}) = \begin{cases} \frac{2n}{n+1} & \text{if n is odd,} \\ \frac{2n}{n+2} & \text{if n is even.} \end{cases}$$

which tends to 2 when n tends to infinity. In contrast:

$$\gamma_1^*(\mathcal{H}) = \begin{cases} \frac{n+1}{2} & \text{if n is odd,} \\ \frac{n+2}{2} & \text{if n is even.} \end{cases}$$

- When n is odd, the family of blocking coalitions of the majority game coincides with the family of winning coalitions. Instead, when n is even, the family of minimal blocking coalitions $\mathcal{H}$ is the family of subsets of cardinality $\frac{n}{2}$ and then $\gamma^*(\mathcal{H}) = 2$ while $\gamma_1^*(\mathcal{H}) = \frac{n+2}{2}$.

Compound Simple games (I)

- Let $(N_r, \mathcal{W}_r)_{1 \leq r \leq R}$ be a family of R hypergraphs with $N_r \cap N_t = \varnothing$ for all $r, t = 1, ..., R$ with $r \neq t$. Let $(N, \mathcal{W})$ be such that $N = \cup_{r=1}^{R} N_r$ and $S \in \mathcal{W}$ iff $S \cap N_r \in \mathcal{W}_r$ for all $r = 1, ..., R$. This is the definition of a multicameral legislature as defined by Diermeier and Myerson (1999): a reform is approved if it is approved in all the different R chambers according to the rules (possibly different) being used in the chambers. It is easy to show that:

$$\gamma^*(\mathcal{B}) = \sum_{r=1}^{R} \gamma^*(\mathcal{B}_r).$$

Compound Simple games (II)

- This multicameral system is a special case of a compound simple game as first defined by Shapley (1962). Let $\left(\{1, ..., R\}, \widetilde{\mathcal{H}}\right)$ be a hypergraph on the set of chambers: $\widetilde{\mathcal{H}}$ describes the power of coalitions of chambers (Diermeier and Myerson (1999)'s definition corresponds to the case where $\widetilde{\mathcal{H}} = \{\{1, ..., R\}\}$, i.e. each chamber has a veto power). In general, $S \in \mathcal{H}$ iff:

$$\{r \in \{1, ..., R\} : S \cap N_r \in \mathcal{W}_r\} \in \widetilde{\mathcal{W}}.$$

- The computation of $\gamma^*(\mathcal{W})$ is now more intricate.

Comments (I)

- In general it is difficult to derive the exact value of $\gamma^*(\mathcal{H})$
- The intersection pattern of winning coalitions plays some role in the determination of the integral and fractional hurdle factors. A cover is a set which intersects every edge. When the simple game is proper, the set of minimal winning coalitions is an intersecting family. Any set in $\mathcal{W}$ is therefore a cover. This implies that the integral covering number is smaller than $Min_{E \in \mathcal{W}} \# E$: *lobby 0 will have to bribe a subset of legislators no larger than the size of the smallest winning coalition*).

Comments (II)

- The knowledge of the integral hurdle factor provides useful information of the smallest size of a group of legislators able to collectively control the legislative process. When it is equal to 1, we have the familiar notion of a vetoer. When the number is equal to k, this means that there is a subset of k legislators which is represented in any winning coalition and that no smaller subset has this property. When the game is strong, the optimal cover is itself a winning coalition: a vetoer is then a dictator.

Nakamura

- The integral hurdle factor is related to another key parameter of a simple game known as the Nakamura number (Nakamura (1978)).
- Let $G = (N, \mathcal{W})$ be a simple game. The Nakamura number of G, is the integer:

$$\nu(G) = \begin{cases} Min_{\mathcal{W}' \subseteq \mathcal{W}} \#\mathcal{W}' \text{ such that: } \cap_{S \in \mathcal{W}'} S = \varnothing, \\ +\infty \text{ if } \cap_{S \in \mathcal{W}} S \neq \varnothing. \end{cases}$$

- **Result 2** : For any simple game

$$\gamma^*(\mathcal{H}) \leq \gamma_1^*(\mathcal{H}) \leq 1 + \frac{(Min\ \#S : S \in \mathcal{H}) - 1}{\nu(G) - 2} \text{ if } \nu(G) \neq \infty$$

and

$$\gamma^*(\mathcal{H}) = \gamma_1^*(\mathcal{H}) = 1 \text{ if } \nu(G) = \infty.$$

Weighted Majority Games (I)

- A simple game is a weighted majority game if there exists a vector $\omega = (\omega_1, \ldots\ldots, \omega_n, q)$ of $(n+1)$)non negative numbers such that a coalition S is in $\mathcal{W}$ iff $\sum_{i \in S} \omega_i \geq q$; $\omega_i > 0$ is the weight attached to legislator i.
- The vector $\omega \equiv (\omega_1, \ldots, \omega_n)$ is called a *representation* of the simple game. The same game may admit several representations.

Weighted Majority Games (II)

- A simple game is *homogeneous* if there exists a representation ω such that $\sum_{i \in S} \omega_i = \sum_{i \in T} \omega_i$ for all $S, T \in \mathcal{W}_m$. This representation is called *the* homogeneous representation of the simple game, as Isbell (1956) has demonstrated that an homogeneous simple game admits a unique (up to multiplication by a constant). The homogeneous representation ω for which $\sum_{i \in N} \omega_i = 1$ is called the homogeneous normalized representation and a homogeneous representation ω for which ω_i is an integer for all $i \in N$ is called an integral representation.

Weighted Majority Games (III)

Nucleolus and Least Core (I)

- Consider an arbitrary cooperative game with transferable utility (N, V) and let $x \in X_n \equiv \{y \in \Re^n_+ : \sum_{i=1}^n y_i = V(N)\}$. Let $\theta(x)$ be the 2^n dimensional vector whose components are the numbers $V(S) - \sum_{i \in S} x_i$ arranged according to their magnitude i.e. $\theta_i(x) \geq \theta_j(x)$ for $1 \leq i \leq j \leq 2^n$. The *nucleolus* of (N, V) is the unique vector $x^* \in X_n$ such that $\theta(x^*)$ is the minimum, in the sense of the lexicographic order, of the set $\{\theta(y) : y \in X_n\}$. The *least core* is the subset of X_n consisting of the vectors x such that $\theta_1(x) = \theta_1(x^*)$. It will be denoted $LC(V, N)$; by construction $x^* \in LC(V, N)$
- To any simple game, we attach the cooperative game with transferable utility (N, V) defined as follows:

$$V(S) = \begin{cases} 1 \text{ if } S \in \mathcal{W}, \\ 0 \text{ if } S \notin \mathcal{W}. \end{cases}$$

Weighted Majority Games (III)

Nucleolus and Least Core (II)

- Only minimal winning coalitions matter in "minimizing" the vector of excesses. The least core consists in the subset of vectors x such that

$$x \in \underset{y \in S_n}{ArgMax} \underset{S \in \mathcal{W}_m}{Min} \sum_{i \in S} y_i,$$

where $S_n \equiv \{y \in \Re^n_+ : \sum_{i=1}^n y_i = 1\}$.

- **Result 3** : $\gamma^*(\mathcal{W}) = \frac{1}{C^*}$ where $C^* \equiv \underset{y \in S_n}{Max} \underset{S \in \mathcal{W}_m}{Min} \sum_{i \in S} y_i$
- The argument also demonstrates that the set of optimal fractional covers of $(N, \mathcal{W})$ is, up to a division by C^*, the least core of the game induced by the simple game. Since the set of optimal fractional covers is, up to the multiplication by W_0, the set of offers to legislators made by lobby 1 at equilibrium, the least core provides a complete characterization of lobby 1's equilibrium behavior.

Weighted Majority Games (IV)

Nucleolus and Least Core (III)

- Questions: Is it simple to calculate the quantity C^* for some particular families of simple games? How the least core looks like, i.e. how are the different legislators treated?
- Peleg (1968) has demonstrated that the normalized homogeneous representation of an *homogeneous strong weighted majority game* $(N, \mathcal{W})$ coincides with the nucleolus x of (N, V). Similarly, the integral representation of the nucleolus (which is well defined) is the minimum representation of the game, i.e. the unique minimal integral representation of the game. Since the nucleolus is an element of the least core, proposition 3, combined with Peleg's result, provides a straightforward way to calculate $\mu^*(\mathcal{W})$ for strong homogeneous weighted majority games.

Weighted Majority Games (V)

Nucleolus and Least Core (IV)

- For instance, the weighted majority game resulting from a legislature with 4 parties where the number of representatives of each party is described by the vector $\omega = (49, 17, 17, 17)$ is exactly the apex game considered before. It is easy to see that the normalized homogeneous representation is here $\left(\frac{2}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}\right)$. It follows that $\gamma^*(\mathcal{W}) = \frac{5}{3}$.

Weighted Majority Games (VI)

Nucleolus and Least Core (V)

- The task is more intricate, however, when the simple game is not homogeneous. Peleg has also proved that the minimal integral representation of the nucleolus is a minimal integral representation of the game if some condition is fulfilled, and has disproved by means of a counterexample of size 12 that the assertion holds true in general. He asks whether this assertion holds true when the simple game has a *minimum* integral representation. This conjecture has been disproved by Isbell (1969) by means of a counterexample of size 19. Therefore, within the class of non homogeneous weighted majority games, the relationship between the nucleolus (and then covering) and the set of minimal representations is less transparent.

Weighted Majority Games (VII)

Nucleolus and Least Core (VI)

- We would like to know which characteristics of legislator i besides α_i determine the "price" of that legislator from the perspective of lobby 1 (in fact, on the market for votes where the two lobbies compete). This is related to the position of i in the set of minimal winning coalitions. In a weighted majority game, we intuitively expect this price to be positively correlated with the weight of the legislator, if not even exactly proportional to that weight. We have just seen that this intuition is correct in the case of a homogeneous weighted majority game (for an appropriate vector of weights), but that the exact relationship between weights and price is less clear otherwise.
- Relationship with Desirability

Example: US Federal Legislative System

House of Representatives (435)
Senate (100)
Vice President
President

- Winning Coalitions:
  1. more than 1/2 of H + 1/2 of S + VP + P
  2. more than 1/2 of H + more than 1/2 of S + P
  3. 2/3 of H and 2/3 of S
- Least Core:

$$\min(435x_1 + 100x_2 + x_3 + x_4)$$
$$\text{s.t. } 218x_1 + 50x_2 + x_3 + x_4 \geq 1$$
$$218x_1 + 51x_2 + x_4 \geq 1$$
$$290x_1 + 67x_2 \geq 1$$

Solution: convex hull of $(0, \frac{1}{67}, \frac{1}{67}, \frac{16}{67})$ and $(0, \frac{1}{67}, 0, \frac{17}{67})$ and $\gamma^* \approx 1.75$

Example: US Federal Legislative System (II)

hurdle factor

- Blocking Coalitions:
  1. $m_1 = 146, m_2 = m_3 = 0,\ m_4 = 1$
  2. $m_1 = 0, m_2 = 34, m_3 = 0,\ m_4 = 1$
  3. $m_1 = 218, m_2 = m_3 = m_4 = 0$
  4. $m_1 = 0, m_2 = 51, m_3 = m_4 = 0$
  5. $m_1 = 0, m_2 = 50, m_3 = 1,\ m_4 = 0$
- **Solution:**

  Least Core: $(\frac{1}{218}, \frac{1}{51}, \frac{1}{51}, \frac{17}{67})$
  and $\gamma^* \approx 4.31$

Example: United Nation Security Council

5 permanent and 10 non permanent members
passage of bill: at least 9 votes in total subject to approval of all permanent members

- representation: $\left(\underbrace{7, \ldots, 7}_{5 \text{ times}}, \underbrace{1, \ldots, 1}_{10 \text{ times}}\right)$, quota $q = 39$

lobby 1 acts to pass reform
$\min(5x_1 + 10x_2)$
s.t. $x_1 \geq 1$ and $7x_2 \geq 1$
Solution: $(1, 1/7)$
and $\gamma^* \approx 6.43$

lobby 0 acts to block reform
$\min(5x_1 + 10x_2)$
s.t. $5x_1 + 4x_2 \geq 1$
Solution: $(1/5, 0)$
and $\gamma^* = 1$

Example: Multicameral Legislature and the Optimal Hurdle Factor (I)

Diermeier and Myerson (1999)

- Diermeier and Myerson's paper aims to determine the optimal hurdle factor of one of the chambers (say the House) in a multiicameral system, given the hurdle factors of the other chambers where optimal means maximizing the expected aggregate amount of bribes received by the members of the house. They assume that W_0 and W_1 are independent and identically distributed random variables and they offer detailed illustrations of the optimization problem in the case where the marginals are either lognormal or uniform.

Example: Multicameral Legislature and the Optimal Hurdle Factor (II)

- It is important to bear in mind that they conduct their analysis under the assumption that there is no uncertainty about which lobby will move first: lobby 1 always moves first. Let t be the sum of the hurdle factors of the other chambers and s be the hurdle factor of the house. Lobby 1 makes offer when $\frac{W_1}{W_0} \geq s + t$. In such a case, the house receives sW_0. When instead $\frac{W_1}{W_0} < s + t$, the house does not receive any transfer. Let $F(s, t)$ be the corresponding expected income of the house. Diermeier and Myerson's central result asserts that the best response $s^*(t)$ of the house, which can be implemented by choosing of an appropriate simple game $(N, \mathcal{W})$, increases as the external hurdle factor t increases.

Example: Multicameral Legislature and the Optimal Hurdle Factor (III)

- In some circumstances, the lobby which wants the status quo to be preserved acts first. If that is the case, the relevant simple game is the dual game and the relevant hurdle factor is the dual hurdle factor. As is demonstrated below, if lobby 0 makes an offer, then the member of the house receives a fraction of the total bribe (in fact the totality) iff their hurdle factor is smaller than the hurdle factor of the other chamber. Consider the case of a bicameral system and let $\widehat{t}$ be the dual hurdle factor of the other chamber and $\widehat{s}$ be the dual hurdle factor of the house.

Example: Multicameral Legislature and the Optimal Hurdle Factor (IV)

- Let $\widehat{F}(\widehat{s}, \widehat{t})$ be the corresponding expected income of the house. If we assume that the two situations occur with probabilities p and $1 - p$ (Diermeier and Myerson assume $p = 1$), then in the simple case where there is no other chamber (unicameral legislature), the expected income is now:

$$pF(s, 0) + (1 - p)F(\widehat{s}, 0)$$

as $F = \widehat{F}$ in the unicameral case.

Example: Multicameral Legislature and the Optimal Hurdle Factor (V)

- A new trade-off appears as increasing s now has two effects: a direct effect like before (as active lobbying becomes less likely) and an indirect effect through a decrease of $\widehat{s}$. Of course, in the above expressions, there is a one to one relationship between the two hurdle factors $\widehat{s} = \gamma^* (\mathcal{W})$ and $s = \gamma^* (\mathcal{B})$. If we limit the implementation to symmetric quota games i.e. $S \in W$ iff $\# S \geq q$, we know that $\widehat{s} = \gamma^* (\mathcal{W}) = \frac{n}{q}$ and $s = \gamma^* (\mathcal{B}) = \frac{n}{n-q+1}$. If n is large, we deduce that:

$$\frac{q}{n}\gamma^* (\mathcal{W}) = \left(1 - \frac{q}{n}\right) \gamma^* (\mathcal{B}) \text{ i.e. } \widehat{s} = \frac{s}{s-1}$$

- First order condition :

$$p\frac{\partial F}{\partial s}(s,0) = \frac{1-p}{(s-1)^2}\frac{\partial F}{\partial s}\left(\frac{s}{s-1},0\right)$$

Example: Multicameral Legislature and the Optimal Hurdle Factor (IV)

The following table provides the value of the optimal hurdle factor for different values of the parameters p and σ in the lognormal case.

Table: Optimal Hurdle Factor in Lognormal Model

p/σ	**0.6**	**0.8**	**1.0**	**1.2**	**1.3**	**1.5**	**1.6**	**1.7**	**2.0**
1	1.0	1.0	1.24	1.85	2.34	3.99	5.38	7.41	21.95
0.75	1.0	1.55	1.69	1.93	2.19	3.53	4.85	6.84	21.33
0.5	2.0	2.0	2.0	2.0	2.0	2.0	3.03	5.41	20.01

Example: Multicameral Legislature and the Optimal Hurdle Factor (V)

- The first line of the table is of course similar to the first line of table 3 in Diermeier and Myerson. An interesting observation is that moving from $p = 1$ to the more balanced assumption $p = \frac{1}{2}$ leads to the optimality of the standard majority game for a large range of values of σ (approximately when σ is less than 1.57).
- Exploiting the symmetry for $p = \frac{1}{2}$, we know that if s is a solution then $\frac{s}{s-1}$ is also a solution. In table 1, we have reported the largest of the two solutions. Interestingly enough, it is larger than Diermeier-Myerson's optimal hurdle factor for small enough values of σ and smaller afterwards. When σ gets larger than 1.57, the optimal hurdle factor increases but stays smaller than Diermeier-Myerson's one.

Example: Multicameral Legislature and the Optimal Hurdle Factor (VI)

- In a multicameral legislature, F's second argument is no longer equal to 0. Given the hurdle factors $\gamma^* (\mathcal{W}_r)$ of each chamber $r = 1, ..., R$,

$$\gamma^* (\mathcal{W}) = Min_{1 \leq r \leq R} \; \gamma^* (\mathcal{W}_r) .$$

- This result has important implications for the determination of the optimal dual hurdle factor by the house. Indeed, in the case where the first-mover lobby is the lobby which wants to block the passage of the reform, the amount of money received by the house will critically depend upon its dual hurdle factor compared to the dual hurdle factors of the other chambers. If it is larger than the smallest one, then the house will not be approached by the lobby.

Example: Multicameral Legislature and the Optimal Hurdle Factor (VII)

- The game describing the interaction between the chambers displays discontinuous payoff functions. In the case of two chambers and $p = \frac{1}{2}$, we obtain that the payoff of chamber 1 is equal to:

$$\begin{cases} \frac{F(\gamma^*(\mathcal{B}_1),\gamma^*(\mathcal{B}_2))}{2} + \frac{\widehat{F}(\gamma^*(\mathcal{W}_1),\gamma^*(\mathcal{W}_2))}{2} & \text{if } \gamma^*(\mathcal{W}_1) < \gamma^*(\mathcal{W}_2), \\ \frac{F(\gamma^*(\mathcal{B}_1),\gamma^*(\mathcal{B}_2))}{2} + \frac{\widehat{F}(\gamma^*(\mathcal{W}_1),\gamma^*(\mathcal{W}_2))}{4} & \text{if } \gamma^*(\mathcal{W}_1) = \gamma^*(\mathcal{W}_2), \\ \frac{F(\gamma^*(\mathcal{B}_1),\gamma^*(\mathcal{B}_2))}{2} & \text{if } \gamma^*(\mathcal{W}_1) > \gamma^*(\mathcal{W}_2), \end{cases}$$

Example: Multicameral Legislature and the Optimal Hurdle Factor (VIII)

- if we assume that ties are broken equally. Interestingly enough, if both chambers were acting under the presumption that the lobby which will move first is the pro-status quo lobby, then the game becomes a Bertrand game where behavioral responses converge to the Nash equilibrium $(1, 1)$. It would be interesting to know what we obtain in the general case. When it is taken for granted that the pro-reform lobby moves first, Diermeier and Myerson found convergence towards the Nash equilibrium $(2.20, 2.20)$ in the case of a bicameral legislature implemented by a quota of 54.5%; note that then $\gamma^* (\mathcal{W}) \simeq 1.835$.

Buying Supermajorities (I)

- Until now, we have assumed that $\alpha = 0$ and we have therefore ignored the impact of vector α on the equilibrium outcomes of the lobbying game. Instead, we have focused our attention on the implications of the rules governing the decision process within the legislature and the "power" derived by the legislators as a result of their status.
- We reintroduce vector α, but we focus our attention on a very special (while important) simple game $(N, \mathcal{W})$ namely the classical majority game and n is odd i.e. $n = 2k - 1$ for some integer $k \geq 2$. In that respect, the analysis of this section is aligned with the framework of Banks (2000) and Groseclose and Snyder (1996)(2000). Given the symmetry of the simple game, all legislators are alike in terms of their power in the legislature. This means that if two legislators i and j receive different offers from the lobby, the rationale for this differential should be based on differences between α_i and α_j.

Buying Supermajorities (II)

- We have seen that some legislators endowed with limited power within the legislature were, sometimes, totally ignored by the lobby. Here, a legislator i with a large α_i will be cheap for lobby 1 and expensive for lobby 0. Finally, we have also observed that most of the time the lobby was bribing a coalition strictly larger than a minimal winning coalition. These considerations raise a number of questions:

Buying Supermajorities (II)

QUESTIONS

- What will be the size of the coalition of legislators receiving offers from the lobby? Since the α_i are nonnegative numbers, lobby 1 may bribe a submajority coalition (at the extreme, nobody at all), a minimal majority or a supermajority (at the other extreme, everybody) depending upon the profile α. Which legislators will be part of that coalition and, in particular, when the cheapest strategy of lobby 1 consists in bribing the whole legislature?
- Which legislators will be part of the bribed coalition? Will we observe a flooded coalition as in Banks (2000) or a nonflooded coalition as in Groseclose and Snyder (1996)(2000), where flooded refers to the fact that lobby 1 bribes in priority the legislators more willing to support the reform.
- What are the differences between the offers received by the legislators who are in the coalition?

Buying Supermajorities (III)

- **Result 4** : If W_1 is large enough, there exists an optimal offer $t_1^* = (t_{11}^*, t_{21}^*, ..., t_{n1}^*)$ by lobby 1 described by an integer $m^* \in \{0, 1, ..., n\}$ and such such that $t_{i1}^* > 0$ and $t_{i1}^* + \alpha^i \Delta W = t_{j1}^* + \alpha^j \Delta W$ for all $i, j = 1, ..., m^*$. Further, either $\frac{W_0}{k} > \alpha^k \Delta W$ and m^* is determined as the unique smallest integer m such that $\frac{W_0}{k} \leq \Delta W \alpha^m$ if any and $m^* = n$ otherwise. Or $\frac{W_0}{k} \leq \alpha^k \Delta W$ and m^* is the smallest value of $m \leq k - 1$ such that: $W_0 < \Delta W \left[\sum_{l=m+1}^{k} \alpha^l + m\alpha^{m+1}\right]$.
- We have exhibited an optimal offer such that all the legislators bribed by lobby 1 end up with an identical net payoff (Groseclose and Snyder call these strategies leveling strategies). There may exist other optimal strategies, even when $m^* > 0$. For example, in the case where $k = 3$, $\alpha^1 = \alpha^2 = \alpha^3 = 0$, $\alpha^4 = \alpha^5 = \beta$ with $W_0 < \beta \Delta W$, we derive easily that any offer $t_1 \in \mathbb{R}_+^5$ such that $\sum_{1 \leq i \leq 3} t_1^i = W_0$ and $t_1^4 = t_1^5 = 0$ is optimal.

Buying Supermajorities (IV)

- How the result answers the three questions formulated at the beginning of the section. Note first that if:

$$\frac{W_0}{k} > \Delta W \alpha_n, \text{ i.e. } \frac{W_1}{W_0} < 1 + \frac{1}{k\alpha^n},$$

then lobby 1's cheapest offer consists in bribing all the legislators. The corresponding cost is $\frac{nW_0}{k} - \Delta W \sum_{l=1}^{n} \alpha^l$ and lobby 1 will therefore find it profitable to do so iff:

$$W_1 \geq \frac{nW_0}{k} - \Delta W \sum_{l=1}^{n} \alpha^l, \text{ i.e. } \frac{W_1}{W_0} \geq \frac{\left(\frac{2k-1}{k}\right) + \sum_{i \in N} \alpha^i}{1 + \sum_{i \in N} \alpha^i},$$

- It is optimal to bribe at least a majority of legislators, it is necessary and sufficient that:

$$\frac{W_0}{k} > \Delta W \alpha^k, \text{ i.e. } \frac{W_1}{W_0} < 1 + \frac{1}{k\alpha^k}.$$

Buying Supermajorities (V)

- It will bribe a minimal majority if:

$$1 + \frac{1}{k\alpha^{k+1}} \leq \frac{W_1}{W_0} < 1 + \frac{1}{k\alpha^k}.$$

- The corresponding cost is $W_0 - \Delta W \sum_{l=1}^{k} \alpha^l$ and lobby 1 will therefore always find it profitable to do so. At the other extreme, if:

$$W_0 < \Delta W \sum_{l=1}^{k} \alpha^l,$$

then, lobby 1 does not offer any bribe.

- While derived under quite different assumptions, Result 4 shares some common features with Banks's main result.

Part 4

SEQUENTIAL COMPETITIVE VOTE BUYING WITH CONSEQUENTIAL LEGISLATORS

Being Pivotal (I)

- A coalition $T \subseteq N$ will be called blocking $+$ if $S = T \backslash \{i\} \in \mathcal{B}_m$ for all $i \in T$. Let us denote by $\mathcal{B}_m^+$ the family of minimal blocking $+$ coalitions.
- Let us examine intuitively the reaction $t_0 = (t_{i0})_{i \in N}$ of lobby 0 to the vector of offers $t_1 = (t_{i1})_{i \in N}$ made by lobby1. The legislators can be partionned into three groups. The first group S_1 consists of the legislators i such that $t_{i0} \leq t_{i1}$. The second group S_2 consists of all the legislators i such that $t_{i1} < t_{i0} \leq t_{i1} + \alpha^i W_1$. The third group S_3 consists of all the legislators i such that $t_{i0} > t_{i1} + \alpha^i W_1$.

Being Pivotal (II)

- Voting for the reform is a dominant strategy for the legislators from the first group while voting for the status quo is a dominant strategy for the legislators in the third group. The strategic interaction and the necessity to evaluate the probability of being pivotal only apply to the legislators from the second group. If a legislator does not consider himself to be pivotal, then it is optimal to vote for the status quo. Instead, if he considers his vote to be pivotal, then it is optimal to vote for the reform.

Being Pivotal (III)

- We want the profile of votes from the legislators in that group to form a Nash equilibrium. Let S be the coalition of legislators being in the second or third group i.e. $S = S_2 \cup S_3$. When is it the case that the profile where all the legislators in S vote for the status quo is a Nash equilibrium ?
- It is necessary and sufficient that no legislator i from S_2 considers his vote to be pivotal. This will be the case if $S \backslash \{i\} \in \mathcal{B}$. Let $\widehat{\mathcal{B}}$ be the family of coalitions $S = S_2 \cup S_3 \subseteq N$ such that $S \backslash \{i\} \in \mathcal{B}$ for all $i \in S_2$ and let $\widehat{\mathcal{B}}_m$ be the family of minimal coalitions in $\widehat{\mathcal{B}}$ according to the order $\lhd$ defined as follows : $(S_2 \cup S_3) \lhd (S_2' \cup S_3')$ iff $S_2 \subseteq S_2'$ and $S_3 \subseteq S_3'$.

Being Pivotal (IV)

- The strategic optimal response of lobby 0 is now easy to describe. From its perspective, the cheapest coalitions belong to the family $\mathcal{B}_m^+$. To any such coalition $S = S_2 \cup S_3$, the smallest cost is equal to:

$$\sum_{i \in S_2} t_{i1} + \sum_{i \in S_3} \left(t_{i1} + \alpha^i \Delta W \right)$$

Being Pivotal (III)

- It is interesting to see what coalitions are part of $\widehat{\mathcal{B}}_m$. First, all the coalitions S in $\mathcal{B}_m$ belong to $\widehat{\mathcal{B}}_m$. They correspond to the case where $S_2 = \varnothing$. Their cost is therefore:

$$\sum_{i \in S} \left(t_{i1} + \alpha^i \Delta W\right)$$

- At the other extreme, all the coalitions S in $\mathcal{B}_m^+$ belong to $\widehat{\mathcal{B}}_m$. They correspond to the case where $S_3 = \varnothing$. Their cost is therefore:

$$\sum_{i \in S} t_{i1}$$

Being Pivotal (IV)

- This reasoning calls for two observations. We note first that in the case where the simple game is symmetric, we obtain: $\widehat{\mathcal{B}}_m = \mathcal{B}_m \cup \mathcal{B}_m^+$. In fact, in such a case, a coalition S is in iff $S = T \cup \{i\}$ for some $i \in N \backslash T$.
- Second, it is important to note that we have determined conditions under which there exists a Nash profile of votes leading to rejection of the reform. This does not mean of course that this Nash equilibrium is unique. The calculation of the cheapest offer is subordinated to the selection of this particular continuation equilibrium which focuses on the worst case from the perspective of lobby 1: following its vector of offers, what is the worst Nash equilibria in the continuation game ?

Being Pivotal (V)

- Let $t_1 = (t_{11}, t_{21}, ..., t_{n1}) \in \Re^n_+$ be lobby 1's offers. Lobby 0 will find profitable to make a counter offer if either there exists a coalition $S = S_2 \cup S_3$ such that:

$$\sum_{i \in S_2} t_{i1} + \sum_{i \in S_3} \left(t_{i1} + \alpha^i \Delta W \right) < W_0$$

The New Linear Program

- If lobby 1 wants to make an offer that cannot be cancelled by lobby 0, it must satisfy the list of inequalities:

$$\sum_{i \in S_2} t_{i1} + \sum_{i \in S_3} \left(t_{i1} + \alpha^i \Delta W\right) \geq W_0 \text{ for all } S = S_2 \cup S_3 \in \widehat{\mathcal{B}}_m$$

- The cheapest offer t_1 meeting these constraints is solution of the following linear program:

$$\underset{t_{1 \in \Re^n_+}}{Min} \sum_{i \in N} t_{i1} \tag{5}$$

subject to the constraints

$$\sum_{i \in S_2} t_{i1} + \sum_{i \in S_3} \left(t_{i1} + \alpha^i \Delta W\right) \geq W_0 \text{ for all } S = S_2 \cup S_3 \in \widehat{\mathcal{B}}_m$$

- Lobby 1 will find profitable to offer the optimal solution t_1^* of problem (1) if the optimal value to this linear program is less than W_1.

The New Linear Program

- If lobby 1 wants to make an offer that cannot be cancelled by lobby 0, it must satisfy the list of inequalities:

$$\sum_{i \in S_2} t_{i1} + \sum_{i \in S_3} \left(t_{i1} + \alpha^i \Delta W \right) \geq W_0 \text{ for all } S = S_2 \cup S_3 \in \widehat{\mathcal{B}}_m$$

- The cheapest offer t_1 meeting these constraints is solution of the following linear program:

$$\underset{t_{1 \in \Re^n_+}}{Min} \sum_{i \in N} t_{i1} \tag{6}$$

subject to the constraints

$$\sum_{i \in S_2} t_{i1} + \sum_{i \in S_3} \left(t_{i1} + \alpha^i \Delta W \right) \geq W_0 \text{ for all } S = S_2 \cup S_3 \in \widehat{\mathcal{B}}_m$$

- Lobby 1 will find profitable to offer the optimal solution t_1^* of problem (1) if the optimal value to this linear program is less than W_1.

The New Dual Linear Program

$$\underset{\gamma}{Max} \sum_{S \in \widehat{\mathcal{B}}_m} \delta(S) \left[W_0 - \sum_{i \in S_3} \alpha^i \Delta W \right]$$

subject to the constraints

$$\sum_{S \in \widehat{\mathcal{B}}_m} \delta(S) \leq 1 \text{ for all } i \in N$$

$$\text{and } \delta(S) \geq 0 \text{ for all } S \in \widehat{\mathcal{B}}_m$$

- When $\alpha = 0$, we are back to the case of procedural voters.

Simple Majority Game with Three Legislators

- $S \in \mathcal{B}_m$ iff $\#S = 2$ i.e. $S = \{1,2\}$, $\{1,3\}$, $\{2,3\}$ and $S \in \mathcal{B}_m^+$ iff $\#S = 3$. Besides the partitions, the unique minimal balanced family of coalitions is $\{\{1,2\}, \{1,3\}, \{2,3\}\}$ with the vector of balancing coefficients$\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$. We deduce that

$$v^*(\mathcal{B}, \alpha) = \ Sup\left(W_0 - (\alpha_1 + \alpha_2)\Delta W, \ \frac{3W_0 - 2(\alpha_1 + \alpha_2 + \alpha_3)\Delta W}{2}\right.$$

$$\gamma^*(\mathcal{B}) = \frac{3}{2}$$

- The first and last terms are never the largest; $v^*(\mathcal{B}, \alpha) = W_0$ whenever $2(\alpha_1 + \alpha_2 + \alpha_3)\Delta W \geq W_0$. When $\alpha_1 = \alpha_2 = \alpha_3 = \alpha$, this happens when:

$$W_1 \geq \frac{1 + 6\alpha}{6\alpha} W_0$$

Per Capita Least Core and Desirability (I)

- We can demonstrate that the set of solutions of the linear program is strongly connected to the per-capita least core of the cooperative game V defined as follows :

$$V(S) = \begin{cases} W_0 - \sum_{i \in S_3} \alpha^i \Delta W \text{ if } S = S_2 \cup S_3 \in \widehat{\mathcal{B}}_m \\ 0 \text{ if } S \notin \widehat{\mathcal{B}}_m \end{cases}$$

- In some cases, it will be possible to order, partially or totally, the legislators according to *desirability* as defined by Maschler and Peleg (1966). Legislator $i \in N$ is at least as desirable as legislator $j \in N$ if $S \cup \{j\} \in \mathcal{W}$ implies $S \cup \{i\} \in \mathcal{W}$ for all $S \subset N \backslash \{i, j\}$. Legislators i and j are symmetric or interchangeable if $S \cup \{j\} \in \mathcal{W}$ iff $S \cup \{i\} \in \mathcal{W}$ for all $S \subset N \backslash \{i, j\}$.

Per Capita Least Core and Desirability (II)

- Legislator i is said to be strictly more desirable than legislator j are symmetric if $S \cup \{j\} \in \mathcal{W}$ implies $S \cup \{i\} \in \mathcal{W}$ for all $S \subset N \setminus \{i, j\}$ and $S \cup \{i\} \in \mathcal{W}$ and $S \cup \{j\} \notin \mathcal{W}$ for some $S \subset N \setminus \{i, j\}$. Examples below shows that two symmetric legislators do not need to be treated similarly in all equilibria of the lobbying game.

More Illustrations (I)

- Simple Games with Large Winning Coalitions (Small Blocking Coalitions)
- When we consider the blocking hypergraph attached to a simple game, the fractional and integral covering numbers are likely to be large numbers when its set of edges contains many small coalitions. This will happen as soon as in the simple game, a coalition is winning if it contains most of the players. The extreme case of such situation is unanimity according to which a coalition is winning if it contains all the legislators. In such case, any singleton is a blocking coalition and then $\mu_1^*(\mathcal{B}) = \gamma^*(\mathcal{B}) = \gamma_1^*(\mathcal{B}) = n$. The closest situation to unanimity is the case where minimal winning coalitions contain either $n-2$ or $n-1$ legislators. This case has been extensively studied by several authors including Lucas (1966), Maschler (1963) and Owen ((1968), (1977).

More Illustrations (II)

- In such a case the set of minimal blocking coalitions consists of a subset of the set of pairs of legislators i.e. the blocking hypergraph $(N, \mathcal{B})$ is an *ordinary graph*. Of course from $(N, \mathcal{B})$, we reconstruct, via duality, the set of winning coalitions $(N, \mathcal{W})$. We obtained that any $S \in \mathcal{W}_m$ is such that either $\#S = n - 1$ or $\#S = n - 2$: $N \setminus \{i\} \in \mathcal{W}_m$ iff $\{i, j\} \in \mathcal{B}$ for all $j \in N$ and $N \setminus \{i, j\} \in \mathcal{W}_m$ iff $\{i, j\} \notin \mathcal{B}$.

More Illustrations (III)

- In such a case, we can take advantage of the results established in the theory of graphs to derive information on the different hurdle factors. In that respect, it will aslo be useful to calculate the matching and fractional matching numbers to obtain lower bounds on the hurdle factor(s). The largest possible value of $\gamma^*(\mathcal{B})$ is $\frac{n}{2}$ which is realized, for instance, when the graph is complete. From the point of view of matchings, it correspond to what is called in graph theory as a *perfect matching*. If there is a perfect matching, $\gamma^*(\mathcal{B}) = \frac{n}{2}$. If the graph is bipartite, Hall's theorem provides necessary and sufficient condition for the existence such a perfect matching. For an arbitrary graph, Tutte's beautiful theorem also provides necessary and sufficient condition for the existence such a perfect matching.
- Reforming the Security Council and the IMF
- Amending the Canadian constitution

Thank you!
See You Soon Maurice